Five-dimensional general relativity and Kaluza-Klein theory

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Abstract

We consider 5D spaces which admit the most symmetric 3D subspaces. 5D vacuum Einstein equations are constructed and 5D analog of the mass function is found. The corresponding conservation law leads to 5D analog of Birkhoff's theorem. Hence the cylinder condition is dynamically implemented for the considered spaces. For some obtained metrics a period of space with respect to the fifth coordinate was found. The problem of the dynamical degrees of freedom of the fields system obtained in the process of dimensional reduction is discussed, and the problem of their interpretation is considered. One can think that the parametrization of the scalar field and 4D metric leading to the conformally invariant 4D theory for interacting gravitational and scalar fields is most natural and adequate.

1 Introduction

The basic proposal of Kaluza-Klein theory is the cylinder condition. However it is one of those assumptions which generates many questions and requires some justification. Other proposal is the idea of compactifying the fifth dimension. There are also some open questions here, one of which is the magnitude of the period of 5D space V^5 with respect to fifth coordinate.

On the other hand in General Relativity the following result is well-known (Birkhoff's theorem): under certain conditions, the Schwarzschild solution is a unique spherically symmetric solution of vacuum Einstein equations [1]. This theorem is in fact the consequence of existence of the mass function and the corresponding conservation law. In 5-dimensional (5D) General Relativity an alleviated version of Birkhoff's theorem [2] is valid: the static, spherically symmetric solution of 5D vacuum Einstein equations is unique [3], but there is a variety of non-stationary vacuum solutions [4].

It turns out that the above mentioned questions are connected. Expanding the concept of the spherical symmetry we can introduce 5D analog of the mass function and formulate 5D analog of Birkhoff's theorem. For this purpose it is enough to perform replacement: $\{V^4, O(2), S^2\} \to \{V^5, O(3), S^3\}$ in the definition of spherical symmetry, where $O(2)$ and $O(3)$ are the rotation groups. They are the groups of motion V^4 and V^5 which are transitive on 2D and 3D spheres S^2 and S^3 respectively. In this meaning we can pose the problem of existence and uniqueness of 5D analog of the Schwarzschild solution. Hence we come to the cylinder condition and to the period of 5D space V^5 with respect to fifth coordinate.

In this paper we consider the more general case, when 5D space V^5 admits a transitive action of the isometry group on 3D spacelike or timelike surfaces Σ^3 of constant curvature. In the framework of 5D General Relativity we construct 5D vacuum Einstein equations, and find 5D analog of the mass function. Hence we obtain the conservation law which points to the fact that 5D analog of the Birkhoff's theorem takes place. Thus we come to the conclusion that in the generalized curvature coordinates all quantities do not depend on the fifth coordinate. The last means that for such spaces the cylinder condition is implemented dynamically. For some obtained metrics we find the period of space with respect to fifth coordinate. The problem of the dynamical degrees of freedom for the system of the interacting scalar and gravitational fields which obtained in the process of dimensional reduction is discussed. The separation problem of dynamical degrees of freedom and their interpretations is considered. The various representations of the obtained metrics are discussed.

2 Birkhoff's theorem in five-dimensional gravity

In the framework of 5D General Relativity [5] let us consider a pseudo-Riemannian space V^5 with the metric ${}^{(5)}g_{AB}$ $(A,\ B = 0,\,1,\,2,\,3,\,4)$ (signature $(+\,-\,-\,-\,-)$) which generally depends on all coordinates x^A. The metric satisfies 5D vacuum Einstein equations

$$ {}^{(5)}G^A_B = {}^{(5)}R^A_B - \frac{1}{2}\ {}^{(5)}R\ \delta^A_B = 0. \tag{1}$$

These equations can be derived by varying 5D version of the usual Einstein-Hilbert action:

$$ I = -\int \sqrt{|{}^{(5)}g|}\ {}^{(5)}R\ d^5x\,, \tag{2}$$

where ${}^{(5)}g = \det||{}^{(5)}g_{AB}||$.

We consider the spaces V^5 which are the generalization of the spherically-symmetric spaces of the General Relativity. These spaces admit the maximally symmetric 3D surfaces Σ^3. For generality, we shall consider both spatial and time-spatially surfaces Σ^3. Thus V^5 admits a transitive action of the isometry group on 3D spacelike or timelike surfaces Σ^3 of constant curvature.

The desired metric can be represented in the following 2+3 form

$$ {}^{(5)}ds^2 = {}^{(5)}g_{AB}dx^A dx^B = g_{ab}dx^a dx^b - \Lambda^2\,{}^{(3)}d\Omega^2\,. \tag{3}$$

where, according to [6] (for more details see Appendix A),

$$ {}^{(3)}d\Omega^2 = h_{ij}dx^i dx^j = \frac{\epsilon(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{\left(1 + \frac{K_0}{4}\,S^2\right)^2} \tag{4}$$

is the metric of 3D space of the unit positive ($K_0 = 1$), negative ($K_0 = -1$) or zero ($K_0 = 0$) curvature. Here $S^2 = \epsilon(x^1)^2 + (x^2)^2 + (x^3)^2$, $\epsilon = 1$ for the spatial section Σ^3 and $\epsilon = -1$ for the time-spatially section Σ^3. The quantities g_{ab} and Λ depend on the coordinates x^a $(a, b = 0, 4;\ \ i, j = 1, 2, 3)$ only.

The components of the Ricci tensor for the metric (3) have the form:

$$
\begin{aligned}
{}^{(5)}R_{ab} &= {}^{(2)}R_{ab} - 3\Lambda^{-1}\nabla_a\nabla_b\Lambda\,, &(5)\\
{}^{(5)}R_{ai} &\equiv 0\,, &(6)\\
{}^{(5)}R_{ik} &= [\Lambda\Delta\Lambda + 2(\nabla\Lambda)^2 + 2K_0]h_{ik}\,, &(7)
\end{aligned}
$$

where $\Delta = \nabla^a\nabla_a$, $(\nabla\Lambda)^2 = g^{ab}\nabla_a\Lambda\nabla_b\Lambda$, ∇_a is the covariant derivative with respect to coordinate x^a calculated with the help of the 2D metric g_{ab}, ${}^{(2)}R_{ab}$ is the Ricci tensor and ${}^{(2)}R$ is the curvature scalar of 2D space with the metric g_{ab}. Hence, according to (1) we obtain the equations of 5D gravitation for the metric (3):

$$
\begin{aligned}
{}^{(5)}G^a_b &= -\frac{3}{\Lambda}\nabla^a\nabla_b\Lambda + \frac{3}{\Lambda^2}\left(\Lambda\Delta\Lambda + (\nabla\Lambda)^2 + K_0\right)\delta^a_b = 0\,, &(8)\\
{}^{(5)}G^i_k &= \left(\frac{1}{2}\Lambda^2\,{}^{(2)}R - 2\Lambda\Delta\Lambda - (\nabla\Lambda)^2 - K_0\right)\delta^i_k = 0\,. &(9)
\end{aligned}
$$

From these equations it follows that

$$
2\Lambda^3\left({}^{(5)}G^a_a\Lambda_{,b} - {}^{(5)}G^a_b\Lambda_{,a}\right) = 3\left(\Lambda^2(\nabla\Lambda)^2 + K_0\Lambda^2\right)_{,b} = 0\,. \tag{10}
$$

Hence one finds the conservation law

$$
G \equiv \Lambda^2(\nabla\Lambda)^2 + K_0\Lambda^2 = \mathrm{const}\,, \tag{11}
$$

which points to the fact that 5D analog of the Birkhoff's theorem takes place. 4D analog of (11) corresponds to the conservation law of the complete mass inside a collapsing ball and determines the mass function [7]. In 5D case it corresponds to the conservation of quantity being the integral characteristic of the sources of some scalar field. Therefore, the quantity G can be named as charging function and corresponds to the conservation law of the scalar charge in 5D General Relativity.

From the equations (1), (7) it follows that $\Lambda\Delta\Lambda + 2(\nabla\Lambda)^2 + 2K_0 = 0$. Hence, using (11) one obtains

$$
\Delta\Lambda + \frac{2G}{\Lambda^3} = 0\,. \tag{12}
$$

Let us consider now the regions in which $\mathrm{sign}(\nabla\Lambda)^2 = \mathrm{sign}\,\epsilon$. Using the admissible transformations $x^a = x^a(\tilde{x}^b)$ one can choose the coordinates $\tilde{x}^b$ so that $\widetilde{g}_{04} = 0$ and $\widetilde{x}^0 = \Lambda$. The new coordinates $\{\widetilde{x}^0 = \Lambda,\ \widetilde{x}^4 \equiv Z\}$ are the analog of the curvature coordinates in General Relativity. As a result 5D interval (3) can be rewritten in the form

$$
{}^{(5)}ds^2 = -M^2dZ^2 + \epsilon N^2 d\Lambda^2 - \Lambda^2\,{}^{(3)}d\Omega^2\,. \tag{13}
$$

Then from the conservation law (11) it follows that

$$
N^{-2} = -\,\epsilon K_0 + \frac{\epsilon\, G}{T^2}\,. \tag{14}
$$

With the help of (12) we find $\partial(MN)/\partial\Lambda = 0$. Hence $M = N^{-1}f(Z)$, where $f(Z)$ is an arbitrary function. One can suppose, without loss of generality, that $f(Z) = 1$. As a result we obtain the metric

$$
{}^{(5)}ds^2 = -\left(-\epsilon K_0 + \frac{\epsilon G}{\Lambda^2}\right)dZ^2 + \frac{\epsilon d\Lambda^2}{-\epsilon K_0 + \frac{\epsilon G}{\Lambda^2}} - \Lambda^2\,\frac{\epsilon(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{\left(1 + \frac{K_0}{4}S^2\right)^2}\,, \tag{15}
$$

which essentially proves 5D Birkhoff's theorem for V^5.

It is easy to see that the obtained metric has the singular null hypersurfaces $\Lambda = \pm\sqrt{K_0 G}$ when $K_0 G > 0$. In some sense they are similar to the Schwarzschild event horizon. When $\epsilon = -1$ the regularity condition of the sections $x^i = \text{const}$ on these horizons leads to the period $L = 2\pi\sqrt{K_0 G}$ of V^5 with respect to fifth coordinate. In the case $\epsilon = 1$, considering the prolongation of the metric on the imaginary axis $Z = \imath Z'$, we obtain the same period for Z'.

In the case of $\epsilon = 1$ setting $\Lambda = T$, we have

$$ {}^{(5)}ds^2 = -\left(-K_0 + \frac{G}{T^2}\right) dZ^2 + \frac{dT^2}{-K_0 + \frac{G}{T^2}} - T^2 \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{\left(1 + \frac{K_0}{4} S_+^2\right)^2} . \tag{16} $$

This metric is 5D analog of the Schwarzschild solution. It admits the rotation group $O(3)$. We can fix the signs K_0 and G by requiring the space V^5 to admit 4D flat sections. As follows from Sec.3 it is possible when $K_0 = -1$ and $G < 0$.

For the sections Σ^3 of positive curvature ($K_0 = 1$) the condition $g_{TT} > 0$ is possible when $G > 0$ and it leads to the finiteness of the model in time $-\sqrt{G} < T < \sqrt{G}$. For the sections Σ^3 of negative curvature ($K_0 = -1$) the metric (16) will be regular for all $T \neq 0$ if the condition $G > 0$ is satisfied. If $G < 0$, the inequality $g_{TT} > 0$ will lead to two domains $T < -\sqrt{-G}$ and $T > \sqrt{-G}$.

The other solutions can be obtained by an analytic continuation the solution (16) through the null hypersurfaces $T = \pm\sqrt{K_0 G}$. In this case the sense of coordinates Z and T varies. Therefore it is necessary to perform the replacement $Z \to T$, $T \to Z$. As a result the dependence of metric on Z appears. This situation is similar to transition through the Schwarzschild horizon. Here we do not consider these regions. The other variant of the metric is possible under replacement $Z \to T$, $T \to R$, $x^1 \to x^4 = z$. Then we come to the metric

$$ {}^{(5)}ds^2 = \left(K_0 - \frac{G}{R^2}\right) dT^2 - \frac{dR^2}{K_0 - \frac{G}{R^2}} - R^2 \frac{(dz)^2 + (dx^2)^2 + (dx^3)^2}{\left(1 + \frac{K_0}{4} S_+^2\right)^2} . \tag{17} $$

It can be transformed to the Tangerlini metric [8]. Here the fifth coordinate and two spatial coordinates are associated by symmetry $O(3)$. Therefore, when a rotation from this group takes place, they are transforming jointly. However, by the data, the fifth coordinate should be transformed in each point V^4 independently on the space-time coordinates as a coordinate of the internal symmetry space. That is why this solution is inconsistent with statement of the problem and we do not also consider it here.

In the case of $\epsilon = -1$ we suppose $\Lambda = R$, $x^1 \to x^0 = t$ and the metric V^5 acquires the form

$$ {}^{(5)}ds^2 = -\left(K_0 - \frac{G}{R^2}\right) dZ^2 - \frac{dR^2}{K_0 - \frac{G}{R^2}} + R^2 \frac{(dt)^2 - (dx^2)^2 - (dx^3)^2}{\left(1 + \frac{K_0}{4} S_-^2\right)^2} . \tag{18} $$

For the sections Σ^3 of positive curvature ($K_0 = 1$) when $G > 0$ the condition $g_{RR} < 0$ performs for the exterior regions $R < -\sqrt{G}$ and $R > \sqrt{G}$ of the space V^5. In case that $G < 0$ the inequality $g_{RR} < 0$ is possible for all R. For the sections Σ^3 of negative curvature ($K_0 = -1$) the condition $g_{RR} < 0$ is possible for the interior region $-\sqrt{-G} < R < \sqrt{-G}$ when $G < 0$. There are the singular null hypersurfaces $R = \pm R_G = \pm\sqrt{K_0 G}$ when $K_0 G > 0$, however transition through these hypersurfaces leads to the non-physical metric.

By analogy with t-independent (static character) Schwarzschild solution in the R-domain, the metrics (16), (18) are independent on the fifth coordinate $x^4 = Z$. Besides, provided $K_0 G > 0$, this coordinate (or its imaginary prolongation for the metric (16)) has the period $L = 2\pi\sqrt{K_0 G}$ for the metric (18). Thus Kaluza's cylinder condition is implemented dynamically here. It explains also why the fifth dimensionality is compact and gives the value of its period.

3 Some representations of obtained solutions

It turns out that the metric (15) admits conformally flat 4D sections. Indeed, after substitution

$$\Lambda = U\left(1 - \frac{\epsilon G}{4U^2}\right), \tag{19}$$

under condition $K_0\epsilon = -1$, the metric is converted as:

$$^{(5)}ds^2 = -\left(\frac{1+\frac{\epsilon G}{4U^2}}{1-\frac{\epsilon G}{4U^2}}\right)^2 dZ^2 + \left(1-\frac{\epsilon G}{4U^2}\right)^2 \left(\epsilon dU^2 - U^2\,{}^{(3)}d\Omega^2\right). \tag{20}$$

In the case $\epsilon G < 0$ formula (19) defines the maps of the exterior region $\sqrt{-\epsilon G} < \Lambda < \infty$ 3-spheres $\Lambda = \sqrt{-\epsilon G}$ onto the region $-\infty < U < \infty$. The metric (20) is invariant with respect to the reflection $U \to -U$ and the inversion $U \to U' = \epsilon G/4U$ of the coordinate U.

With the help of the formulas (43), (46), where we suppose $\epsilon_{\mu\nu} = -\eta_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$), $e \to \epsilon$, $\Lambda \to U$, the interval (20) can be rewritten as

$$^{(5)}ds^2 = -\left(\frac{1+\frac{\epsilon G}{4U^2}}{1-\frac{\epsilon G}{4U^2}}\right)^2 dZ^2 + \left(1-\frac{\epsilon G}{4U^2}\right)^2 \eta_{\mu\nu}dy^\mu dy^\nu, \tag{21}$$

where $U^2 = \eta_{\mu\nu}y^\mu y^\nu$, and $\eta_{\mu\nu}$ is the Minkowski metric. The metric (21) is invariant with respect to conformal transformations which in addition to the Lorentz transformation contain inversion operation:

$$y^\mu = \frac{\epsilon G}{4}\frac{y'^\mu}{U'^2}, \qquad (U'^2 = \eta_{\alpha\beta}y'^\alpha y'^\beta).$$

When $K_0\epsilon = 1$, from the condition $g_{ZZ} < 0$, it follows that $\epsilon G > 0$. Then after the substitutions

$$\Lambda = \sqrt{\epsilon G}\,\cos\ln\left(\frac{U}{\sqrt{\epsilon G}}\right), \quad \Lambda = \sqrt{\epsilon G}\,\sin\ln\left(\frac{U}{\sqrt{\epsilon G}}\right) \tag{22}$$

the metric (15) can be written in the forms

$$^{(5)}ds^2 = -\tan^2\ln\left(\frac{U}{\sqrt{\epsilon G}}\right) dZ^2 + \frac{\epsilon G}{U^2}\cos^2\ln\left(\frac{U}{\sqrt{\epsilon G}}\right)\left(\epsilon dU^2 - U^2\,{}^{(3)}d\Omega^2\right), \tag{23}$$

$$^{(5)}ds^2 = -\cot^2\ln\left(\frac{U}{\sqrt{\epsilon G}}\right) dZ^2 + \frac{\epsilon G}{U^2}\sin^2\ln\left(\frac{U}{\sqrt{\epsilon G}}\right)\left(\epsilon dU^2 - U^2\,{}^{(3)}d\Omega^2\right). \tag{24}$$

respectively.

For the case $G < 0$ and $K_0\epsilon = -1$ the metric (15) admits also flat 4D space-time slices which are non-orthogonal to the coordinate lines $x^5 = Z$. Indeed, it may be rewritten in the form similar to Painlevé representation [9] of the Schwarzschild solutions. This can be performed with the help of transformation of the fifth coordinate

$$Z = z + \int \left(\frac{\Lambda}{\Lambda_G} - \epsilon \frac{\Lambda_G}{\Lambda} \right)^{-1} d\Lambda\,, \tag{25}$$

where $\Lambda_G = \sqrt{-G}$. As a result the metric (15) can be rewritten as

$$^{(5)}ds^2 = -dz^2 + \epsilon \left(d\Lambda - \epsilon \frac{\Lambda_G}{\Lambda}\, dz \right)^2 - \Lambda^2\, {}^{(3)}d\Omega^2\,. \tag{26}$$

At last, with help of the formulas (43), (46) of appendix where $\epsilon_{\mu\nu} = -\eta_{\mu\nu}$, $e = \epsilon$, the interval (26) can be written in the form

$$^{(5)}ds^2 = -dz^2 + \eta_{\mu\nu} \left(dy^\mu - \epsilon \frac{\Lambda_G}{\Lambda}\, \eta^\mu dz \right) \left(dy^\nu - \epsilon \frac{\Lambda_G}{\Lambda}\, \eta^\nu dz \right)\,. \tag{27}$$

Here $\Lambda^2 = \epsilon \eta_{\mu\nu} y^\mu y^\nu$, $\eta^\mu = y^\nu/\Lambda$, $d\Lambda = \epsilon \eta_\mu dy^\mu$. Hence it can be seen that in the coordinates under consideration 4D physical space is the set of flat space-time sections of a normal geodesic congruence of the curves in V^5 with a field of tangential vectors $U^A = \{U^5 = 1,\ U^\mu = \epsilon(\Lambda_G/\Lambda)\eta^\mu\}$. It also follows from (27) that the singularity $\Lambda = \Lambda_G$ in the metric (15) is stipulated by a choice of the coordinates and is associated with incompleteness of the curvature coordinates $\{Z, \Lambda\}$ similarly to the singularity of the event horizon $R = R_g$ in Schwarzschild metric. However, in contrast to the curvature singularity $R = 0$ of the Schwarzschild solution, the curvature singularity $\Lambda = 0$ is situated on the light cone $(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0$.

Note that in case of the sections Σ^3 of zero curvature ($K_0 = 0$) the metric (16) has other simple representations. Taking into account the condition $g_{ZZ} < 0$, we suppose $G > 0$. Then, after replacement $T = x^0\sqrt{G}$, this metric becomes

$$^{(5)}ds^2 = -(x^0)^{-2} dZ^2 + G(x^0)^2 \left((dx^0)^2 - (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right)\,. \tag{28}$$

In case of the metric (18) for $K_0 = 0$ it is necessary to put $G < 0$. Then after replacement $G = -\widetilde{G},\ R \to x^1\sqrt{\widetilde{G}},\ x^1 \to x^0$ we have

$$^{(5)}ds^2 = -(x^1)^{-2} dZ^2 + \widetilde{G}(x^1)^2 \left((dx^0)^2 - (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right)\,. \tag{29}$$

These metrics are 5D analogs of Kasner metric (the degenerate case). After the change $x^0 = 2\sqrt{\tau\tau_0}$, $Z = 2\tau_0 y$ and $G = (2\tau_0)^{-2}$ expression (28) coincide with metric of the cosmological model considered in [10].

4 On the separation of dynamical degrees of freedom

In order to separate the space-time dynamical variables from dynamical variable of the interior space in some solutions of 5D gravitation one must construct (4+1)-split of 5D metric and perform the relevant transformation of 4D metric and scalar field resulting from dimensional reduction. There are no special problems at the first step, since the split methods are well known (for example, see [11] and references therein) and metric is sufficiently ordinary. Further, a problem of a conformal rescaling or conformal gauge is emerged. It is associated with conformal ambiguity of the physical metric and scalar field on V^4 and with a possibility of their conformal transformation.

In case of V^5 under consideration the dynamical system is in the state with one conserved nonzero quantity only. In configurational space it corresponds to one dynamical degree of freedom which can be connected either with gravitational field or with some scalar field. Therefore we shall consider the basic metric ansatzs of the Kaluza-Klein theory from this point of view.

Let us rewrite the interval (3) in (4+1)-form

$$ {}^{(5)}ds^2 = g_{\mu\nu} dx^\mu dx^\nu - W^2 dz^2 \,, \tag{30} $$

As it was shown above, for the considered symmetric case the cylinder condition is implemented dynamically. Therefore quantities W and $g_{\mu\nu}$ does not depend on the coordinate z, and hence for 5D action (2) we have

$$ I = -L \int \sqrt{-{}^{(4)}g}\; W\, {}^{(4)}R \; d^4x \,, \tag{31} $$

where L is the period of V^5 with respect to fifth coordinate, ${}^{(4)}R$ is the scalar curvature of 4D space with metric $g_{\mu\nu}$.

We are free to perform the transformation

$$ W = W(\varphi), \quad g_{\mu\nu} = f(\varphi)\widetilde{g}_{\mu\nu} \,. $$

Therefore the metric of the physical space-time is determined up to a conformal factor. The transformation should be chosen so that it was possible “to orthogonalize” in some sense the action (31) and to separate the degrees of freedom corresponding to the scalar and gravitational field. One of the most popular ansatzs (see, for example, [12] and [13]) is

$$ {}^{(5)}ds^2 = e^{\varphi/\sqrt{3}} \widetilde{g}_{\mu\nu} dx^\mu dx^\nu - e^{-2\varphi/\sqrt{3}}\, dz^2 \,. \tag{32} $$

In terms of new variables the action (31) reduces to the action for the gravitational field $\widetilde{g}_{\mu\nu}$ and the scalar field φ with minimal coupling

$$ I = -L \int d^4x \sqrt{-{}^{(4)}g} \left\{ {}^{(4)}\widetilde{R} - \frac{1}{2} \widetilde{g}^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} \right\} \,. \tag{33} $$

Comparing together the metrics (32) and (16) we obtain the scalar field and the new metric

$$ \varphi = -\frac{\sqrt{3}}{2} \ln \left(-K_0 + \frac{G}{T^2} \right) \,, \quad \widetilde{g}_{\mu\nu} = g_{\mu\nu} \sqrt{-K_0 + \frac{G}{T^2}} \,, \tag{34} $$

where the metric $g_{\mu\nu}$ can be obtained from the solution (16). The new fields φ and $\widetilde{g}_{\mu\nu}$ are functionally dependent and therefore can not represent the independent dynamical degrees of freedom of the system. On that ground we should reject the ansatz (32) as far as it does not correspond to the optimal representation of the dynamical system with one degree of freedom.

Now we consider the other ansatz [13]

$$^{(5)}ds^2 = \left(1+\frac{\psi}{\sqrt{6}}\right)^2 \,{}^{(4)}ds'^{\,2} - \left(\frac{1-\frac{\psi}{\sqrt{6}}}{1+\frac{\psi}{\sqrt{6}}}\right)^2 dz^2\,. \tag{35}$$

For this case the expression (31) leads to the action

$$^{(4)}I = -L\int d^4x\sqrt{-{}^{(4)}g}\left\{\left(1-\frac{\psi^2}{6}\right)R' - g'^{\,\mu\nu}\psi_{,\mu}\psi_{,\nu}\right\}\,, \tag{36}$$

which describes the interacting gravitational $g'_{\mu\nu}$ and conformally invariant scalar ψ fields. The equations of motion for new system have the form

$$(\Delta - \tfrac{1}{6}R')\psi = 0\,, \tag{37}$$

$$G'_{\mu\nu} = 4\pi t_{\mu\nu} \equiv 4\pi T_{\mu\nu} + \tfrac{1}{6}\left(G'_{\mu\nu} - \nabla_\mu\nabla_\nu + g'_{\mu\nu}\Delta\right)\psi^2\,, \tag{38}$$

$$4\pi T_{\mu\nu} = \psi_\mu\psi_\nu - \tfrac{1}{2}\,g'_{\mu\nu}(\nabla\psi)^2\,, \tag{39}$$

where $t_{\mu\nu}$ is the conformally invariant energy-momentum tensor of scalar field ψ with conformal coupling, $G'_{\mu\nu} = R'_{\mu\nu} - \frac{1}{2}\,g'_{\mu\nu}\,R'$, $R'_{\mu\nu}$ and R' are the Ricci tensor and the curvature scalar respectively, $\Delta = \nabla^\mu\nabla_\mu$, ∇_μ is the covariant derivative with respect to the coordinate x^μ. Here all the quantities are calculated with the help of 4D metric $g'_{\mu\nu}$.

The metrics (35) and (21) have the same form. Hence, as a result of the comparison, we obtain the scalar field and the physical metric

$$\psi = -\frac{\sqrt{6}}{4}\,\frac{\epsilon\, G}{(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2}\,,\qquad g'_{\mu\nu} = \eta_{\mu\nu}\,. \tag{40}$$

It is interesting that the conformal energy-momentum tensor vanishes $t_{\mu\nu} = 0$ for the obtained solution of the equations (37-39). The received representation connects the single degree of freedom of the system (which associated with conserved charge G) with the conformally invariant scalar field ψ. It corresponds to the condition of a separating problem of degrees of freedom. Here the physical space-time is flat, and the scalar field ψ is the classical ghost. It has zero energy density and does not render influence on the space-time.

It is easy to generalize the considered ansatz to a case of the electromagnetic field A_μ [13] when instead of (30) we use the metric

$$^{(5)}ds^2 = g_{\mu\nu}dx^\mu dx^\nu - W^2(dz + A_\mu dx^\mu)^2\,. \tag{41}$$

Gauge transformations $z = z' + f(x^\mu)$, $A_\mu = A'_\mu - f_{,\mu}$ leave this metric invariant.

Now let us appeal to the metrics (26) and (27). They are already written in (4+1)-form and obeys the separating condition of the dynamical degrees of freedom. Here constant (Λ_G) enters neither the scalar field $(\varphi = 0)$ nor the 4D metric $(g_{\mu\nu} = \eta_{\mu\nu})$. It appears in 5D shift vector $U^A = \{U^5 = 1,\ U^\mu = \epsilon(\Lambda_G/\Lambda)\eta^\mu\}$. However these metrics are not gauge-invariant with respect to the above transformations and should be rejected.

If we use the more general (4+1)-split V^5 with a nonholonomic basis [11], it will be possible to write the metric (27) (for example for $\epsilon = 1$) as

$$ {}^{(5)}ds^2 = -(\theta^5)^2 + \eta_{\mu\nu}\left(dx^\mu - U^\mu(\theta^5)\right)\left(dx^\nu - U^\nu(\theta^5)\right) , \tag{42} $$

where $\theta^5 = dz + A_\mu dx^\mu$, $U^\mu = \eta^\mu T_G/T$ and $A_\mu = f_{,\mu}$. From this extended standpoint the metric (42) satisfies the above conditions and can be considered as one of the models in 5D General Relativity. The scalar, electromagnetic and gravitational fields are absent here, but instead there is the shift vector field U^μ. The physical space-time is represented by set of flat hypersurfaces V^4 which are embedded in the curved space V^5 with non-vanishing exterior curvature.

5 Conclusion

In the paper it is shown that for the spaces of 5D General Relativity, which admit the most symmetric 3D subspaces, the cylinder condition is implemented dynamically. Moreover there is a thing which can be termed as a "spontaneous compactification" of the fifth dimension. The regularity condition of the metric (18) leads to a closure of V^5 by the fifth coordinate Z with the period $L = 2\pi\sqrt{K_0 G}$ where $K_0 G > 0$. We use the parametrization of 5D metric such as (35) which is used in the conformally invariant theory of interacting the scalar and gravitational fields (36). We suppose that such reduction of this solution to 4D form is most natural and adequate to real physics. The conformally invariant representation (20) is possible for $K_0 = 1$, which gives $L = 2\pi\sqrt{G}$ and $G > 0$.

Thus radius $R_G = \sqrt{G} = L/2\pi$ of the event horizon of 5D black hole with the metric (18), plays a role of the fundamental scale of the theory. After the conformal transformation $\Lambda = U(1 + G/4U^2)$ the metric takes the form (20) with $\epsilon = -1$. The system has one classical ghost degree of freedom which is associated to scalar field (40) on the background of the flat physical metric $\eta_{\mu\nu}$.

The Kaluza-Klein theory can be understood in some sense as a limit case of the spaces of 5D General Relativity admitting the most symmetric 3D subspaces. We consider the obtained solution as a ground nontrivial state of 5D geometry. We can treat its asymmetric perturbations generated by the classical or quantum excitations as an induced matter [5]. However before proceeding further, it is necessary to study a stability V^5 with respect to small perturbations like $\delta g_{AB}(x^\mu)\exp nx^5$ by analogy with Schwarzschild metric.

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A On the metrics of constant curvature spaces

Let us consider the flat space E^{n+1} with metric:

$$ {}^{(n+1)}ds^2 = \epsilon_{\alpha\beta}dy^\alpha dy^\beta \quad (\alpha,\, \beta \,=\, 1, \ldots , n+1)\,, \tag{43} $$

where $\epsilon_{\alpha\beta} = c_\alpha \delta_{\alpha\beta}$ (there is no summation here!). According to [6] the basic hypersurfaces of the second order

$$\epsilon_{\alpha\beta} y^\alpha y^\beta = e\Lambda^2 \tag{44}$$

were $e = \pm 1$, represent the single hypersurfaces S^n of the constant curvature $K = e/\Lambda^2$ of the space E^{n+1} with the metric (43).

The family of hypersurfaces (44) (where a parameter of the family is Λ) induces $(n+1)$-decomposition of E^{n+1} and of all objects on it [11]. Let us introduce the field of the unit vectors

$$n^\alpha = \frac{y^\alpha}{\Lambda}, \qquad \epsilon_{\alpha\beta} n^\alpha n^\beta = e\,. \tag{45}$$

Then

$${}^{(n+1)}ds^2 = ed\Lambda^2 + {}^{(n)}ds^2\,, \qquad \epsilon_{\alpha\beta} = e\, n_\alpha n_\beta + h_{\alpha\beta} \tag{46}$$

where

$${}^{(n)}ds^2 = h_{\alpha\beta} dy^\alpha dy^\beta\,, \qquad d\Lambda = e\, n_\alpha dy^\alpha\,. \tag{47}$$

Here ${}^{(n)}ds^2$ is the metric on the hypersurfaces $\Lambda = \text{const}$ which are the hypersurfaces of the constant curvature $K = e/\Lambda^2$.

Now we introduce the standard normalized hypersurface S_0^n with the coordinates

$$z^\alpha = n^\alpha = \frac{y^\alpha}{\Lambda}, \qquad \epsilon_{\alpha\beta} z^\alpha z^\beta = e. \tag{48}$$

Then

$${}^{(n+1)}ds^2 = ed\Lambda^2 + \Lambda^2 d\Omega^2\,, \tag{49}$$

where

$$d\Omega^2 = \epsilon_{\alpha\beta} dz^\alpha dz^\beta \qquad (\epsilon_{\alpha\beta} z^\alpha z^\beta = e)\,, \tag{50}$$

is the metric for the space of fixed curvature $K_0 = e$. Let us eliminate from here the coordinate z^{n+1}. We have

$$z^{n+1} = \sqrt{c_{n+1}(K_0 - S_z^2)}\,, \qquad S_z^2 \equiv \epsilon_{ik} z^i z^k \quad (i,\, k \;=\; 1, \ldots, n)\,, \tag{51}$$

$$d\Omega^2 = g_{ik} dz^i dz^k\,, \tag{52}$$

where

$$g_{ik} = \epsilon_{ik} + \frac{c_{n+1} z^i z^k}{(z^{n+1})^2} \tag{53}$$

is the metric for the space S_0^n of curvature $K_0 = e$, and $\epsilon_{ik} = c_i \delta_{ik}$ (there is no summation here).

As a result of stereographic projection of S_0^n onto E^n

$$z^i = x^i \left(1 + \frac{K_0}{4} S_x^2\right)^{-1} \qquad (S_x^2 \equiv \epsilon_{ik} x^i x^k)\,, \tag{54}$$

we obtain the following new metric tensor for S_0^n

$$\widetilde{g}_{kl} = g_{ik} \frac{\partial z^i}{\partial x^k} \frac{\partial z^j}{\partial x^l} = \epsilon_{kl} \left(1 + \frac{K_0}{4} S_x^2\right)^{-2}\,. \tag{55}$$

Finally we can write for the metric of the constant curvature space $K = e/\Lambda^2$

$$ {}^{(n)}ds^2 = {}^{(n+1)}ds^2 - ed\Lambda^2 = (\epsilon_{\alpha\beta} - e\, n^\alpha n^\beta) dy^\alpha dy^\beta = \Lambda^2 d\Omega^2 \,, \tag{56} $$

where

$$ d\Omega^2 = \frac{\epsilon_{kl} dx^k dx^l}{\left(1 + \frac{K_0}{4} S_x^2\right)^2} \tag{57} $$

is the metric of space S_0^n of the unit curvature $K_0 = e = \pm 1$ and $\epsilon_{\alpha\beta} n^\alpha n^\beta = e$. In addition we note

$$ y^i = \frac{\Lambda x^i}{1 + \frac{K_0}{4} S_x^2} \,, \qquad y^{n+1} = \Lambda \sqrt{c_{n+1} K_0}\, \frac{1 - \frac{K_0}{4} S_x^2}{1 + \frac{K_0}{4} S_x^2} \qquad \left(S_x^2 = \epsilon_{ik} x^i x^k\right) \,. \tag{58} $$

Hence it can be seen that the above stereographic projection S_0^n is possible when $c_{n+1} = K_0$.

References

[1] D. Kramer, H. Stephani, M.Maccallum and E. Herlt, Exact Solution of the Einstein Field Equations (Energoizdat, Moskow, 1982).

[2] K.A. Bronikov and V.N Melnikov, Gen. Rel. & Grav. 27 (1995) 465.

[3] D. Kramer, Acta Phys. Pol. B 2 (1970) 807.

[4] S.S. Kokarev and V.G. Krechet, Grav. & Cosmol. 2 (1996) 107.

[5] J.M. Overduin, P.S. Wesson, Phys. Rep. 283 (1997) 303.

[6] L.P. Eisenhart, Riemannian Geometry (Princeton University, Princeton, 1926).

[7] W.C. Hernandes and G.C. Misner, Astrph. J. (1968) 143 452;
M.E. Cahill and G.C. McVittie, J. Math. Phys. 11 (1970) 1382.

[8] F.R. Tangerlini, Nuovo Cimento 27 (1963) 636.

[9] P. Painlevé, C. R. Acad. Sci. (Paris) 173 (1921) 677.

[10] A. Chodos and S. Detweiler, Phys. Rev. D 21 (1980) 2167.

[11] V.D. Gladush, Acta Phys. Pol. B 30 (1999) 3;
V.D. Gladush, R.A. Konoplya, J. Math. Phys. 40 (1999) 955.

[12] M.J. Duff, Kaluza-Klein Theory in Perspective, hep-th/9410046, 1994.

[13] V.D.Gladush, Izv. Vyssh. Uchebn. Zaved. Fiz 11 (1979) 58
(English translation: Sov. Phys. J. 22 (1979) 1172).